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Unified approaches for construction of PT-symmetric quasi-exactly solvable potentials

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Abstract : Unified approaches in the light of supersymmetric quantum mechanics (SSQM) have been suggested for generating one dimensional PT-symmetric quasi-exactly solvable (PTQES) singular and non-singular potentials, which are new. These PTQES potentials are constructed with the help of Kustaanheino-Steifel transformation of the co-ordinate.

Keywords : SUSY algebras, construction of PTQES potentials

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1. Introduction

In quantum mechanics, the Hamiltonian of a given system is usually Hermitian ensuring a real energy spectrum. Recently, PT-symmetric quantum mechanics proposed by Bender and Boettcher[1] suggested a new picture which is based on a complexification of coordinates which breaks the Hermiticity of Hamiltonian but does not destroy the reality of the energies. Hence, the Schrödinger Hamiltonian can be invariant under the joint action of parity (P) and time reversal (T) transformation. This continuation of real Hamiltonians to complex plane with preservation of their PT-symmetry opens a new and unexplored field of mathematical analysis of the Schrödinger equation. So, this new quantum theory is an extension of conventional Hermitian quantum mechanics into the complex domain. The concept of PT-symmetry is discussed in details in ref. 1 and 2. An overwhelming number of evidences supporting the PT-symmetry are available in Refs.[1-5] and references quoted there in.

Exact solution of Schrödinger equation provides all information about the system concerned. But for physical systems, exactly solvable potentials are very few in number.

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Therefore, the quasi-exactly solvable (QES) potentials have received wide attention [6-8]. These QES models allow exact solutions only for a limited part of the energy spectrum but not for the entire spectrum. Thus, these potentials fill up the gap between the exactly solvable and non-solvable potentials and help to understand many physical phenomena *e.g.* structural phase transitions, polaron formation in solids and the concept of false vacuo in field theory in different fields of physics [9]. Moreover, QES problem has its own inner mathematical beauty as it can provide a good starting point for doing calculations perturbetively for complex systems.

Recently, Bender and Boetcher [10] discovered a large two parametric family of non-Hermitian and PT-symmetric forces which are quasi-exactly solvable. Bagchi *et al* [6,11,12] also have studied the QES PT-symmetric sexit, non-polynomial oscillators and hyperbolic potentials within the frame work of SSQM [13].

Here, we have developed a unified approach for the construction of PTQES potentials of singular and non-singular types inspired by the study of two and three particle models [14] in one dimension. In SSQM, the shape invariant potentials and non shape invariant potentials can be constructed by using different forms of nodeless wavefunctions. In this article, with the help of SSQM, we have constructed generalized one dimensional PTQES potentials which are new, using different forms of the nodeless as well as PT-symmetric wavefunctions. We have also generated new types of PTQES singular and non- singular potentials from our generalized PTQES potentials for different choices of parameters which are exactly solved for the ground state only. The ground state energy being real in each case.

In this article, several aspects of SUSY algebras releted to our problems are reviewed in Section 2 and applications of these algebras to construct generalized form of PTQES potentials in one dimension are discussed in Section 3. Finally, in Section 4, the conclusion has been drawn.

2. SUSY algebras

Consider a particle of mass m moving in one dimension. The time independent Schrödinger equation is given by (in units of $\hbar$=2m = 1)

$$H\Psi = \xi\Psi \tag{1}$$

here, the wavefunction ψ belongs to the energy eigen value ξ. We can write Eq. (1) as

$$H\Psi_1^n(x) = \left[-\frac{d}{dx^2} + V(x)\right]\psi_1^n(x) = \xi_1^n\psi_1^n(x), \tag{2}$$

where the superscripts n refers a quantam number, the interpretation of which depends on the choice of potential $V(x)$.

In SSQM, the Hamiltonian corresponding to the Schrödinger equation (2) can be written in the form

$$H = H_1 = -\frac{d^2}{dx^2} + V_1(x). \tag{3}$$

Here $\xi_1^{(0)} = 0$ is the binding energy of the lowest bound state of H_1 such that

$$H_1\psi_1^{(0)}(x) = 0 \tag{4}$$

The superscript '(0)' on wavefunction ψ stands for the ground state wavefunction while the subscript '1' merely indicates that the wavefunction $\psi_1^{(0)}(x)$ belongs to H_1. The superscript and subscripts on ξ have similar meaning. Here, we shall use analogous notation for the partner Hamiltonians. The parent Hamiltonian operator in eq.(3) can be factorized as

$$H_1 = 0^- 0^+, \tag{5}$$

where the operators

$$0^{\pm} = \pm\frac{d}{dx} + W(x). \tag{6}$$

The supersymmetric partner Hamiltonian of H_1 is

$$H_2 = 0^+ 0^- = -\frac{d^2}{dx^2} + V_2(x). \tag{7}$$

The parent potential $V_1(x)$ and its SUSY partner potential $V_2(x)$ can be written in terms of so-called superpotential $W(x) = -\frac{d}{dx}\ln\Psi_1^{(0)}(x)$ as

$$V_{1,2}(x) = W^2(x) \mp \frac{dW(x)}{dx}. \tag{8}$$

Equation (5) through Eq.(4) automatically guarantees that the ground state of H_1 has zero energy and provides the ground state wavefunction of H_1 as

$$\psi_1^{(0)}(x) = N_0 \exp\left(-\int^x W(x')dx'\right). \tag{9}$$

Here, N_0 is the normalizaion constant.

3. Construction of PTQES potentials

Any function $F(x)$ of a complex argument x is PT-symmetric if

$$[F(x)]^* = F(-x^*). \tag{10}$$

So, any real function of ix is manifestly PT-symmetric. In general, for any PT-symmetric Hamiltonian H, we must solve the Sturm-Liouville differential equation eigenvalue problem associated *with* H as eq.(2), *i.e.*

$$H\psi^n(x) = \xi^n\psi^n(x); n = 0,1,2,3,\ldots \tag{11}$$

This differential equation (11) must be imposed on an infinite contour in the complex x-plane. The boundary conditions on the eigenfunctions are $\psi(x) \to 0$ exponentially rapidly as $|x| \to \infty$ on the contour.

In order to construct the generalized form of PTQES, the algebras of SSQM have been used. Here, we also generate the superpartners of the constructed PTQES potentials which also share the PT-symmetric properties. This procedure is based on the construction of a superpotential *W(x)* which is logarithmic derivative of the well behaved nodeless wavefunction $\psi_1^{(0)}(x)$ on the real axis. For our construction of different types of PTQES potentials, we shall use the different forms of nodeless ground state wavefunctions which are parameterized by two parameters (α,β) and also PT-symmetric. Hence, there are infinitely many PTQES potentials which are parameterized by α and β. For a given set of parameters α and β values, one can construct a number of different types of complex QES potentials which are also PT-symmetric and have real energies.

Kustaanhèimo-Steifel transformation :

Now, we shall discuss the process for construction of PTQES potentials by a small complex shift of the coordinate axis like $x \to x - i\epsilon, x \in (-\infty,\infty)$ which is known as Kustaanheimo-Steifel transformation [15].

Type A :

Here, we first choose the ground state wavefunction like

$$\psi_1^{(0)}(x) = \exp\left(-\alpha(x-i\epsilon)^2 - \beta\sum_{n=0}^{m}\binom{m}{n}(-i)^n x^{m-n}\epsilon^n\right), m \geq 3. \tag{12}$$

This ground state wavefunction is PT-symmetric if and only if m is even number. For odd m, this function is not PT-symmetric and produces a complex potential with complex eigenvalues. So, to construct PT-symmetric potential for odd m, it is necessary to replace the parameter β by $i\beta$.

For even m, using the gound state wavefunction of eq.(12), we obtain the generalized form of PTQES potentials from eq.(8) as

$$V_{1,2}^{m=even}(x) = a(x^2-\epsilon^2) - 2ia\epsilon x + $$
$$+b\sum_{n=0}^{2m-2}\binom{2m-2}{n}(-i)^n x^{2m-n-2}\epsilon^n + c\sum_{n=0}^{m}\binom{m}{n}(-i)^n x^{m-n}\epsilon^n \pm d\sum_{n=0}^{m-2}\binom{m-2}{n}(-i)^n x^{2m-n-2}\epsilon^n \pm e, \tag{13}$$

where the coupling parameters are $a = 4\alpha^2$, $b = \beta^2 m^2$, $c = -4\alpha\beta m$, $d = -\beta m(m-1)$ and $e = -2\alpha$.

Similarly, when m is odd, we obtain from eq.(12), the PT-symmetric ground state wavefunction as

$$\psi_1^{(0)}(x) = \exp\left(-\alpha(x-i\epsilon)^2 - i\beta\sum_{n=0}^{m}\binom{m}{n}(-i)^n x^{m-n}\epsilon^n\right). \tag{14}$$

With the help of this gound state wavefunction, we obtain from eq.(8), another generalized form of PTQES potentials

$$V_{1,2}^{m=odd}(x) = \alpha(x^2-\epsilon^2) - 2ia\epsilon x + b\sum_{n=0}^{2m-2}\binom{2m-2}{n}(-i)^n x^{2m-n-2}\epsilon^n$$
$$+ic\sum_{n=0}^{m}\binom{m}{n}(-i)^n x^{m-n}\epsilon^n \pm id\sum_{n=0}^{m-2}\binom{m-2}{n}(-i)^n x^{2m-n-2}\epsilon^n \pm e, \tag{15}$$

where the coupling parameters are $a = 4\alpha^2$, $b = -\beta^2 m^2$, $c = -4\alpha\beta m$, $d = -\beta m(m-1)$ and $e = -2\alpha$. Now, for a given vaue of parameters α, β and for even and odd values of m, one can construct a number of different types of non-singular complex PTQES potentials which have real ground state spectra. As for example, for $m=3$ and 4, PTQES potentials are given in Table1.

Type B:

Here, we consider the ground state wavefunction like

$$\psi_1^{(0)}(x) = \exp\left(-\alpha\sum_{n=0}^{2m}\binom{2m}{n}(-i)^n x^{2m-n} \in^n -\beta\sum_{n=0}^{m}\binom{m}{n}(-i)^n x^{m-n}\in^n\right). \quad (16)$$

This ground state wavefunction is PT-symmetric if and only if m is even number. For odd m, this function is no PT-symmetric and produces the complex potentials with complex eigenvalue. To construct PT-symmetric new potentials for odd m, it is necessary to replace the parameter β by $i\beta$.

Using the above ground state wavefunction given in Eq.(16) for even values of m, we obtain generalized form of PTQES potentials as

$$V_{1,2}^{m=even}(x) = a\sum_{n=0}^{4m-2}\binom{4m-2}{n}(-i)^n x^{4m-n-2} \in^n +b\sum_{n=0}^{3m-2}\binom{3m-2}{n}(-i)^n x^{3m-n-2} \in^n$$

$$\pm c\sum_{n=0}^{2m-2}\binom{2m-2}{n}(-i)^n x^{2m-n-2} \in^n \pm d\sum_{n=0}^{m-2}\binom{m-2}{n}(-i)^n x^{2m-n-2} \in^n, \quad (17)$$

where the coupling parameters are $a = 4\alpha^2m^2, b = -4\alpha\beta m^2, c = \beta^2m^2 - 2\alpha m(2m-1)$ and $d = \beta m(m-1)$.

To construct PTQES potentials for odd values of m, we must use the ground state wavefunction as

$$\psi_1^{(0)}(x) = \exp\left(-\alpha\sum_{n=0}^{2m}\binom{2m}{n}(-i)^n x^{2m-n} \in^n -i\beta\sum_{n=0}^{m}\binom{m}{n}(-i)^n x^{m-n} \in^n\right). \quad (18)$$

For this case, the generalized form of PTQES potentials are obtained from eq.(8) as

$$V_{1,2}^{m=odd}(x) = a\sum_{n=0}^{4m-2}\binom{4m-2}{n}(-i)^n x^{4m-n-2} \in^n +ib\sum_{n=0}^{3m-2}\binom{3m-2}{n}(-i)^n x^{3m-n-2} \in^n$$

$$\pm c\sum_{n=0}^{2m-2}\binom{2m-2}{n}(-i)^n x^{2m-n-2} \in^n \pm id\sum_{n=0}^{m-2}\binom{m-2}{n}(-i)^n x^{2m-n-2} \in^n, \quad (19)$$

where the coupling parameters are $a = 4\alpha^2m^2, b = -4\alpha\beta m^2, c = -\beta^2m^2 - 2\alpha m(2m-1)$ and $d = \beta m(m-1)$. For a given value of α and β, one can generate a number of different types of potentials for even and odd values of m which have real ground state eigen values. Different types of PTQES potentials are shown in Table 1 for m=3 and 4 only.

Type C :

For this case, we consider the ground state wavefunction

$$\psi_1^{(0)}(x) = \exp\left(-\alpha\sum_{n=0}^{m}\binom{m}{n}(-i)^n x^{m-n} \in^n -\beta\sum_{n=0}^{m+1}\binom{m+1}{n}(-i)^n x^{m-n+1} \in^n\right). \quad (20)$$

With the help of this ground state wavefunction and from eq.(8), we obtain PTQES potentials as

This ground state wavefunction is not PT-symmetric for any value of m but will be so under two conditions. One is for even m with β replaced by $i\beta$ and the other is for odd m with α replaced by $i\alpha$.

For even m, we get PT-symmetric ground state wavefunction as

$$\psi_1^{(0)}(x)=\exp\left(-\alpha\sum_{n=0}^{m}\binom{m}{n}(-i)^n x^{m-n}\in^n -i\beta\sum_{n=0}^{m+1}\binom{m+1}{n}(-i)^n x^{m-n+1}\in^n\right) \tag{21}$$

With the help of this ground state wavefunction and from eq.(8), we obtain PTQES potentials as

$$V_{1,2}^{m=even}(x)=a\sum_{n=0}^{2m}\binom{2m}{n}(-i)^n x^{2m-n}\in^n +ib\sum_{n=0}^{2m-1}\binom{2m-1}{n}(-i)^n x^{2m-n-1}\in^n$$
$$+c\sum_{n=0}^{2m-2}\binom{2m-2}{n}(-i)^n x^{2m-n-2}\in^n \pm id\sum_{n=0}^{m-1}\binom{m-1}{n}(-i)^n x^{m-n-1}\in^n \pm e\sum_{n=0}^{m-2}\binom{m-2}{n}(-i)^n x^{m-n-2}\in^n, \tag{22}$$

where the coupling parameters are $a=-\beta^2(m+1)^2, b=2\alpha\beta m(m+1),\ c=\alpha^2m^2,\ d=-\beta m(m+1)$ and $e=-\alpha m(m-1)$.

Now, for odd values of m, the PT-symmetric ground state wavefunction is

$$\psi_1^{(0)}(x)=\exp\left(-i\alpha\sum_{n=0}^{m}\binom{m}{n}(-i)^n x^{m-n}\in^n -\beta\sum_{n=0}^{m+1}\binom{m+1}{n}(-i)^n x^{m-n+1}\in^n\right). \tag{23}$$

Using eqs. (23) and (8), we obtain the generalized form of PTQES potentials

$$V_{1,2}^{m=odd}(x)=a\sum_{n=0}^{2m}\binom{2m}{n}(-i)^n x^{2m-n}\in^n +ib\sum_{n=0}^{2m-1}\binom{2m-1}{n}(-i)^n x^{2m-n-1}\in^n$$
$$+c\sum_{n=0}^{2m-2}\binom{2m-2}{n}(-i)^n x^{2m-n-2}\in^n \pm d\sum_{n=0}^{m-1}\binom{m-1}{n}(-i)^n x^{m-n-1}\in^n \pm ie\sum_{n=0}^{m-2}\binom{m-2}{n}(-i)^n x^{m-n-2}\in^n, \tag{24}$$

where the coupling parameters are $a=\beta^2(m+1), b=2\alpha\beta m(m+1), c=-\alpha^2m^2,\ d=-\beta m(m+1)$ and $e=-\alpha m(m-1)$. For a given set of α and β and for even and odd values of m, one can generate a number of PTQES potentials which have real ground state spectra. The PTQES potentials for $m=3$ and 4 are tabulated in Table 1.

Type D:

Here, we consider the ground state wavefunction like

$$\psi_1^{(0)}(x)=\exp\left(-\alpha\sum_{n=0}^{2m}\binom{2m}{n}(-i)^n x^{m-n}\in^n -\beta\sum_{n=0}^{m+2}\binom{m+2}{n}(-i)^n x^{m-n+2}\in^n\right), m\geq 3. \tag{25}$$

This ground state wavefunction is not PT-symmetric of any value of m. This will be PT-symmetric for even m and for odd m with α replaced by $i\alpha$ and β replaced by $i\beta$.

With the help of eqs.(25) and (8), we obtain the generalized form of PTQES potentials for even m as

$$V_{1,2}^{m=even}(x)=a\sum_{n=0}^{2m+2}\binom{2m-2}{n}(-i)^n x^{2m-n+2}\in^n +b\sum_{n=0}^{2m}\binom{2m}{n}(-i)^n x^{2m-n}\in^n$$
$$+c\sum_{n=0}^{2m-2}\binom{2m-2}{n}(-i)^n x^{2m-n-2}\in^n \pm d\sum_{n=0}^{m}\binom{m}{n}(-i)^n x^{m-n}\in^n, \pm e\sum_{n=0}^{m-2}\binom{m-2}{n}(-i)^n x^{m-n-2}\in^n \tag{26}$$

where the coupling parameters are $a = \beta^2(m+2)^2, b = 2\alpha\beta m(m+2), c = \alpha^2 m^2, d = -\beta(m+2)(m+1)$ and $e = -\alpha m(m-1)$.

For odd m, the PT-symmetric ground state wave function is

$$\psi_1^{(0)}(x) = \exp\left(-i\alpha\sum_{n=0}^{m}\binom{m}{n}(-i)^n x^{m-n}\epsilon^n - i\beta\sum_{n=0}^{m+2}\binom{m+2}{n}(-i)^n x^{m-n+2}\epsilon^n\right), m \geq 3. \quad (27)$$

Using eqs. (27) and (8), we get the generalized form of PTQES potentials as

$$V_{1,2}^{m=odd}(x) = a\sum_{n=0}^{2m+2}\binom{2m+2}{n}(-i)^n x^{2m-n+2}\epsilon^n + b\sum_{n=0}^{2m}\binom{3m}{n}(-i)^n x^{2m-n}\epsilon^n + c\sum_{n=0}^{2m-2}\binom{2m-2}{n}(-i)^n x^{2m-n-2}\epsilon^n$$

$$\pm id\sum_{n=0}^{m-2}\binom{m}{n}(-i)^n x^{m-n}\epsilon^n \pm ie\sum_{n=0}^{m-2}\binom{m-2}{n}(-i)^n x^{m-n-2}\epsilon^n, \quad (28)$$

where the coupling parameters are $a = \beta^2(m+2)^2, b = 2\alpha\beta m(m+2), c = \alpha^2 m^2$, $d = -\beta(m+2)(m+1)$ and $e = -\alpha m(m-1)$. So, for a given set of parameters, α, β and for even and odd values of m, one can construct a number of different types of non-singular complex QES potentials having real ground state energies. Different types of PTQES potentials for m=3 and 4 are given in Table 1.

Table 1. Different types of PTQES potentials with ground state energies for m=3 and 4.

Type of potentials	Value of m	Value of potential parameters	PTQES potentials	Ground state energy
Type A	$m=4$ $\epsilon \neq 0$ $\alpha, \beta = $ real	$A = 16\beta^2, B = -16\alpha\beta - 240\beta^2\epsilon^2$ $C = 4\alpha^2 \mp 12\beta + 240\beta^2\epsilon^4 + 96\alpha\beta\epsilon^2$, $P = -96\beta^2\epsilon, Q = 320\beta^2\epsilon^3 + 64\alpha\beta\epsilon$, $R = -96\beta^2\epsilon^5 - 64\alpha\beta\epsilon^3$	$V_{1,2}(x) = Ax^6 + Bx^4$ $+Cx^2 + i(Px^5 + Qx^3)$ $+Rx)$	$\xi_{1,2}^{(0)} =$ $16\beta^2\epsilon^6$ $+16\alpha\beta\epsilon^4$ $+4\alpha^2\epsilon^2$ $\mp 12\beta\epsilon^2$ $\pm 2\alpha$
Type A:	$m=3$ $\epsilon \neq 0$ $\alpha = $ real $\beta \to i\beta$	$A = -9\beta^2, B = 4\alpha^2 + 54\beta^2\epsilon^2 - 36\beta\epsilon$ $P = -12\alpha\beta + 36\beta^2\epsilon, Q = -36\beta^2\epsilon^3$ $+36\alpha\beta\epsilon^2 - 8\alpha^2\epsilon \mp 6\beta$	$V_{1,2}(x) = Ax^4 + Bx^2$ $+i(Px^3 + Qx)$	$\xi_{1,2}^{(0)} =$ $9\beta^2\epsilon^4$ $-12\alpha\beta\epsilon^3$ $+4\alpha^2\epsilon^2$ $\pm 6\beta\epsilon$ $\pm 2\alpha$
Type B:	$m=4$, $\epsilon \neq 0$, $\alpha, \beta = $ real	$A = 64\alpha^2, B = -5824\alpha^2\epsilon^2$ $C = 64064\alpha^2\epsilon^4 - 64\alpha\beta, D =$ $-192192\alpha^2\epsilon^6 + 2880\alpha\beta\epsilon^2, E =$	$V_{1,2}(x) = (Ax^{14} + Bx^{12}$ $+Cx^{10} + Dx^8$ $+Ex^6$	$\xi_{1,2}^{(0)} =$ $64\alpha^2\epsilon^{14}$ $-64\alpha\beta\epsilon^{10}$

Table 1. *(Contd.)*

Type of potentials	Value of m	Value of potential parameters	PTQES potentials	Ground state energy
		$192192\alpha^2\epsilon^8 - 13440\alpha\beta\epsilon^4 + 16\beta^2$ $-56\alpha, F = -6406\alpha^2\epsilon^{10} + 13440\alpha\beta\epsilon^6$ $\mp 240\beta^2\epsilon^2 \pm 75\alpha\epsilon^2, G = 5824\alpha^2\epsilon^{12}$ $-2880\alpha\beta\epsilon^8 + 240\beta^2\epsilon^4 - 840\alpha\epsilon^4 \pm 12\beta,$ $P = -896\alpha^2\epsilon, Q = 23296\alpha^2\epsilon^3, R =$ $-128128\alpha^2\epsilon^5 + 640\alpha\beta\epsilon, S = 219648\alpha^2\epsilon^7$ $-7680\alpha\beta\epsilon^3, T = -128128\alpha^2\epsilon^9 +$ $16128\alpha\beta\epsilon^5 \mp 96\beta^2\epsilon \pm 336\alpha\epsilon.$ $U = 23296\alpha^2\epsilon^{11} - 7680\alpha\beta\epsilon^7 \mp 320\beta^2\epsilon^3$ $\pm 1120\alpha\epsilon^3, Z = -896\alpha^2\epsilon^{13} + 640\alpha\beta\epsilon^9$ $\mp 96\beta^2\epsilon^5 \pm 336\alpha\epsilon^5 \mp 24\beta\epsilon.$	$+Fx^4 + Gx^2\big) +$ $i(Px^{13} + Qx^{11} + Rx^9$ $+Sx^7 + Tx^5 + Ux^3 + Zx)$	$\pm 16\beta^2\epsilon^6$ $\mp 56\alpha\epsilon^6$ $\pm 12\beta\epsilon^2$
Type B:	$m=3$, $\epsilon \neq 0$, $\alpha =$ real, $\beta \to i\beta$	$A = 36\alpha^2, B = -1620\alpha^2\epsilon^2, C = 7560\alpha^2\epsilon^4$ $-252\alpha\beta\epsilon, D = \mp 9\beta^2 \mp 30\alpha - 7560\alpha^2\epsilon^6$ $+1260\alpha\beta\epsilon^3, E = 1620\alpha^2\epsilon^8 - 756\alpha\beta\epsilon^5$ $\pm 54\beta^2\epsilon^2 \pm 180\alpha\epsilon^2, P = -360\alpha^2\epsilon,$ $Q = 4320\alpha^2\epsilon^3 - 36\alpha\beta, R = -9072\alpha^2\epsilon^5$ $+756\alpha\beta\epsilon^2, S = 4320\alpha^2\epsilon^7 - 1260\alpha\beta\epsilon^4$ $\pm 36\beta^2\epsilon \pm 120\alpha\epsilon, T = \pm 6\beta \mp 36\beta^2\epsilon^3$ $\mp 120\alpha\epsilon^3 + 252\alpha\beta\epsilon^6 - 360\alpha^2\epsilon^9$	$V_{1,2}(x) = \big(Ax^{10} + Bx^8$ $+Cx^6 + Dx^4 + Ex^2\big)$ $+i\big(Px^9 + Qx^7 +$ $Rx^5 + Sx^3 + Tx\big)$	$\xi^{(0)}_{1,2} =$ $36\alpha^2\epsilon^{10}$ $-36\alpha\beta\epsilon^7$ $\pm 9\beta^2\epsilon^4$ $\pm 30\alpha\epsilon^4$ $\mp 6\beta\epsilon$
Type C:	$m = 4$, $\epsilon \neq 0$, $\alpha =$ real, $\beta \to i\beta$	$A = -25\beta^2, B = 700\epsilon^2 + 280\alpha\beta\epsilon +$ $16\alpha^2, C = -1750\beta^2\epsilon^4 - 1400\alpha\beta\epsilon^3 - 240\alpha^2\epsilon^2,$ $D = 700\beta^2\epsilon^6 + 840\alpha\beta\epsilon^5 + 240\alpha^2\epsilon^4 \mp 60\beta\epsilon$ $\mp 12\alpha, P = 200\beta^2\epsilon + 40\alpha\beta, Q = -1400\beta^2\epsilon^3$ $-840\alpha\beta\epsilon^2 - 96\alpha^2\epsilon, R = 1400\beta^2\epsilon^5 + 1400$ $\alpha\beta\epsilon^4 + 320\alpha^2\epsilon^3 \mp 20\beta, S = -200\beta^2\epsilon^7 -$ $280\alpha\beta\epsilon^6 - 96\alpha^2\epsilon^5 \pm 60\beta\epsilon^2 + 24\alpha\epsilon.$	$V_{1,2}(x) = \big(Ax^8 + Bx^6$ $+Cx^4 + Dx^2\big)$ $+i\big(Px^7 + Qx^5 +$ $Rx^3 + Sx\big).$	$\xi^{(0)}_{1,2} =$ $25\beta^2\epsilon^8$ $+16\alpha^2\epsilon^6$ $\mp 20\beta\epsilon^3$ $\mp 12\alpha\epsilon^2$ $-40\alpha\beta\epsilon$
Type C:	$m=3$, $\epsilon \neq 0$, $\alpha \to i\alpha$, $\beta =$ real	$A = 16\beta^2, B = -240\beta^2\epsilon^2 + 120\alpha\beta\epsilon$ $-9\alpha^2, C = 240\beta^2\epsilon^4 - 240\alpha\beta\epsilon^3 +$ $54\alpha^2\epsilon^2 \mp 12\beta, P = 24\alpha\beta - 96\beta^2\epsilon,$ $Q = 320\beta^2\epsilon^3 - 240\alpha\beta\epsilon^2, R = -96\beta^2$ $+120\alpha\beta\epsilon^4 \pm 24\beta\epsilon \mp 6\alpha.$	$V_{1,2}(x) = \big(Ax^6 + Bx^4$ $+Cx^2\big) + i\big(Px^5$ $+Qx^3 + Rx\big).$	$\xi^{(0)}_{1,2} =$ $16\beta^2\epsilon^6$ $-24\alpha\beta\epsilon^5$ $+9\alpha^2\epsilon^4$ $\mp 12\beta\epsilon^2$ $\pm 6\alpha\epsilon$

Table 1. *(Contd.)*

Type of poten-tials	Value of m	Value of potential parameters	PTQES potentials	Ground state energy
Type D :	$m = 4$, $\in \neq 0$, $\alpha \to i\alpha$, $\beta \to i\beta$	$A = 36\beta^2, B = 48\alpha\beta - 1620\beta^2 \in^2, C = 16\alpha^2 + 7560\beta^2 \in^4 - 1344\alpha\beta \in^2, D = -30\beta - 7560\beta^2 \in^6 - 240\alpha^2 \in^2 + 3360\alpha\beta \in^4, E = 1620\beta^2 \in^8 - 1344\alpha\beta \in_6 + 240\alpha^2 \in^4 \pm 180\beta \in, P = -360\beta^2 \in, Q = 4320\beta^2 \in^3 - 384\alpha\beta \in, R = -9072\beta^2 \in^5 + 2688\alpha\beta \in^3 - 96\alpha^2 \in, S = 4320\beta^2 \in^7 - 2688\alpha\beta \in^5 + 320\alpha^2 \in^3 \pm 120\beta \in, T = -360\beta^2 \in^9 + 384\alpha\beta \in^7 - 96\alpha^2 \in^5 \mp 120\beta \in^3 \pm 24\alpha\beta$	$V_{1,2}(x) = (Ax^{10} + Bx^8 + Cx^6 + Dx^4 + Ex^2) + i(Px^9 + Qx^7 + Rx^5 + Sx^3 + Tx)$.	$\xi_{1,2}^{(0)} = 36\beta^2 \in^{10} - 48\alpha\beta \in^8 + 16\alpha^2 \in^6 \pm 30\beta \in^4 \mp 12\alpha \in^2$
Type D :	$m = 3$, $\in \neq 0$, $\alpha \to i\alpha$, $\beta \to i\beta$	$A = -25\beta^2, B = -15\alpha\beta + 700\beta^2 \in^2, C = -9\alpha^2 - 1750\beta^2 \in^4 + 225\alpha\beta \in^2, D = 700\beta^2 \in^6 - 225\alpha\beta \in^4 + 54\alpha^2 \in^2, P = 200\beta^2 \in^8, Q = -1400\beta^2 \in^3 + 90\alpha\beta \in, R = 1400\beta^2 \in^5 - 300\alpha\beta \in^3 + 36\alpha^2 \in \mp 20\beta, S = -200\beta^2 \in^7 + 54\alpha^2 \in^5 - 36\alpha^2 \in^3 \pm 60\beta \in^2 \mp 6\alpha$.	$V_{1,2}(x) = (Ax^8 + Bx^6 + Cx^4 + Dx^2) + i(Px^7 + Qx^5 + Rx^3 + Sx)$.	$\xi_{1,2}^{(0)} = 25\beta^2 \in^8 + 15\alpha\beta \in^6 - 9\alpha^2 \in^4 \mp 20\beta \in^3 \pm 6\alpha \in$

Type E :

Here, we consider the ground state wavefunction as

$$\psi_1^{(0)}(x) = \exp\left(-\alpha \sum_{n=0}^{m} \binom{m}{n} (-i)^n x^{m-n} \in^n - \frac{\beta}{(x^2+2)^m} \sum_{n=0}^{m} \binom{m}{n} (-i)^n x^{m-n} \in^n\right), m \geq 1. \tag{29}$$

This ground state wavefunction is not PT-symmetric for any value of m. This will be PT-symmetric for the even values of m and as well as for the odd values of m with α replaced by $i\alpha$ and β by $i\beta$.

With the help of eqs. (29) and (8), we obtain the generalized form of the PTQES potentials for even m as

$$V_{1,2}^{m=\text{even}}(x) = a \sum_{n=0}^{2m-2} \binom{2m-2}{n} (-i)^n x^{2m-n-2} \in^n \pm b \sum_{n=0}^{m-2} \binom{m-2}{n} (-i)^n x^{m-n-2} \in^n + \frac{c}{(x^2+\in^2)^2} (x^2 - \in^2 + 2i \in x)$$
$$+ \frac{d}{(x^2+\in^2)^{2m+2}} \sum_{n=0}^{2m+2} \binom{2m+2}{n} (-i)^{2m-n+2} \in^n \pm \frac{e}{(x^2+\in^2)^{m+2}} \sum_{n=0}^{m+2} \binom{m+2}{n} (-i)^n x^{m-n+2} \in^n, \tag{30}$$

where the coupling parameters are $a = \alpha^2 m^2 . b = -\alpha m(m-1), c = -2\alpha\beta m^2, d = \beta^2 m^2$ and $e = -\beta m(m+1)$

For odd m, the PT-symmetric ground state wavefunction is

$$\psi_1^{(0)}(x) = \exp\left(-i\alpha\sum_{n=0}^{m}\binom{m}{n}(-i)^n x^{m-n} \in^n - i\frac{\beta}{\left(x^2+\in^2\right)^m}\sum_{n=0}^{m}\binom{m}{n}(i)^n x^{m-n} \in^n\right), m \geq 1. \tag{31}$$

With the help of eqs. (31) and (8) we obtain the generalized form of the PTQES potentials as

$$V_{1,2}^{m=odd}(x) = a\sum_{n=0}^{2m-2}\binom{2m-2}{n}(-i)^n x^{2m-n-2} \in^n \pm ib\sum_{n=0}^{m-2}\binom{m-2}{n}(-i)^n x^{m-n-2} \in^n$$
$$+\frac{c}{\left(x^2+\in^2\right)^2}\left(x^2-\in^2+2i \in x\right)+\frac{d}{\left(x^2+\in^2\right)^{2m+2}}\sum_{n=0}^{2m+2}\binom{2m+2}{n}(i)^{2m-n+2} \in^n$$
$$\pm i\frac{e}{\left(x^2+\in^2\right)^{m+2}}\sum_{n=0}^{m+2}\binom{m+2}{n}(i)^n x^{m-n+2} \in^n, \tag{32}$$

where the coupling parameters are $a = -\alpha^2 m^2, b = -\alpha m(m-1), c = -2\alpha\beta m^2, d = -\beta^2 m^2$ and $e = -\beta m(m+1)$. So, for a given set of parameters α, β and the even and odd vaues of m, one can generate a number of different types of singular complex PTQES potentials having real ground state energies. The PTQES potentials are given in Table 2 for m=1 and m=2.

Table 2. Different types of PTQES potentials with ground state energies for m =1 and m=2

Type of poten-tials	Value of m	Value of potentials parameters	PTQES potentials	Ground state energy
Type	m = 2,	$a = -4\alpha^2, b = -2\alpha,$	$V_{1,2}(x) = ax^2 + \frac{c\left(x^2-\in^2\right)}{\left(x^2+\in^2\right)^2} + \frac{d}{\left(x^2+\in^2\right)^6}$	$\xi_{1,2}^{(0)} =$
E :	$\in \neq 0,$	$c = -8\alpha\beta,$		$2\alpha(2\alpha \in$
	$\alpha \to i\alpha,$	$d = 4\beta^2, e = -6\beta$	$\left(x^6 - 15 \in^2 x^4 + 15 \in^4 x^2 - \in^6\right) \pm$	$\pm 1)$
	$\beta \to i\beta$		$\frac{e}{\left(x^2+\in^2\right)^4}\left(x^4 - 6 \in^2 x^2 + \in^4\right) + i(-2a \in x$	
			$+\frac{2c \in x}{\left(x^2+\in^2\right)^2} + \frac{d}{\left(x^2+\in^2\right)^6}\left(6 \in x^5 - 20 \in^3 x^3\right.$	
			$\left.+6 \in^5 x \pm \frac{e}{\left(x^2+\in^2\right)^4}\left(4 \in x^3 - 4 \in^3 x\right)\right).$	

Table 2. *(Contd.)*

Type of potentials	Value of m	Value of potentials parameters	PTQES potentials	Ground state energy
Type E :	m = 1	$a=-\alpha^2, b=0,$	$V_{1,2}(x) = \frac{c(x^2-\epsilon^2)}{(x^2+\epsilon^2)^2} + \frac{d}{(x^2+\epsilon^2)^4}$	$\xi^{(0)}_{1,2} =$
	$\epsilon \neq 0,$	$c=2\alpha\beta, d=-\beta^2,$	$(x^4-6\epsilon^2 x^2+\epsilon^4) \pm \frac{e}{(x^2+\epsilon^2)^3}(x^3-3\epsilon x^2$	α^2
	$\alpha \to i\alpha,$	$e=-2\beta,$	$+\epsilon^3) + i(\frac{2c\epsilon x}{(x^2+\epsilon^2)^2} + \frac{d}{(x^2+\epsilon^2)^4}(4\epsilon x^3 -$	
	$\beta \to i\beta$		$\epsilon^3 x) \pm \frac{e}{(x^2+\epsilon^2)^3}(-3\epsilon^2 x)).$	

4. Conclusion

In this paper, we have presented results from supersymmetric point of view. We have constructed a number of new complex PT-symmetric QES potentials in one dimension with the help of a suitable ansatz for the PT-symmetric wavefunctions and shown the associated ground state energy to be real.

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